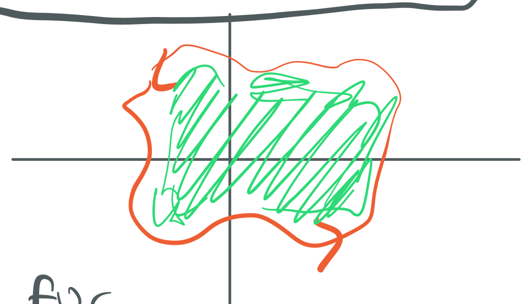
In Class 11/12/2021

Last time:

<u>Greene's Thm</u>: Suppose you know D is a region in the plane w/ boundary a piecewise-smooth SCC. If $P(x,y)$ and $Q(x,y)$ have cts partials on some open region R, containing all of D:

$$\boxed{\int_{\partial D} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA}$$

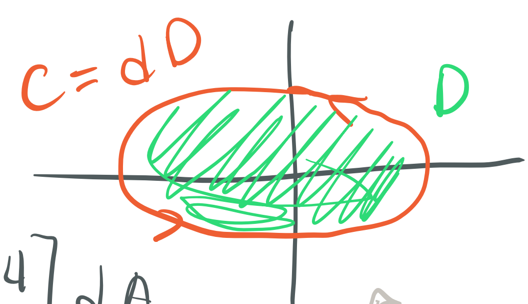


• ex: Compute

$\int_C y^4\,dx + 2xy^2\,dy$ for

C the positively ellipse $x^2+y^2=2$

<u>Sol</u>:



$$\int y^4\,dx + 2xy^2\,dy$$

$$= \iint_D \frac{\partial}{\partial x}[2xy^2] - \frac{\partial}{\partial y}[y^4]\,dA$$

$$= \iint_D 2y^2 - 4y^3\,dA$$

need to parameterize D $\hookrightarrow 0 \le \theta \le 2\pi$

$2x^2 + 2y^2 = 2$

↑ would be circle so need to substitute

$x = \sqrt{2}\, r\cos\theta$

$y = r\sin\theta$

brings in that 2 !! and makes polar

* plug into equation

$$2r^2\cos^2\theta + 2r^2\sin^2\theta = 2$$

$$\hookrightarrow 2r^2 = 2$$

$$0 \le r \le 1 \quad \text{UNIT CIRCLE}$$

$$\int_0^{2\pi}\int_{r=0}^{1} 2y^2 - 4y^3 \quad \sqrt{2}\, r \; dr\, d\theta$$

← Jacobian

Inner:

$$2\sqrt{2}r^3 \int_0^{2\pi} \sin^2\theta(1-2r\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} (1-\cos^2\theta)(1-2r\sin\theta)$$

$$\frac{d(x,y)}{d(r,\theta)} = \det\begin{bmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}\cos\theta & -\sqrt{2}r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$r\sqrt{2}\cos^2\theta + r\sqrt{2}\sin^2\theta$$

$$= \sqrt{2}r$$

↓

$$\int_0^{2\pi} (1-\cos^2\theta)\, d\theta - 2r\int_0^{2\pi} (1-\cos^2\theta)\sin\theta\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right) d\theta + 2r \left[u - \frac{1}{3}u^3\right]_0^{2\pi}$$

$$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) + 2r\left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0) - \frac{1}{4}\left(\cancel{\sin 4\pi}^{0} - \cancel{\sin 0}^{0}\right) + 2r\left(\cancel{\cos 2\pi - \cos 0}^{0} \;\; \cancel{- \tfrac{1}{3}\cos^3 2\pi + \tfrac{1}{3}\cos^3 0}^{0}\right)$$

$$= \pi$$

$$\int_{r=0}^{1} 2\sqrt{2}\, r^3 \pi \, dr$$

$$= \frac{2\sqrt{2}\pi}{4} r^4 \Big|_0^1 = \boxed{\frac{\pi\sqrt{2}}{2}}$$

If P, Q satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,

then by Greenes thm:

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \underline{\text{Area}(D)}$$

$\Downarrow$

can compute w/ line integral!

ex: Compute A of gen. ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol: $\text{Area}(D) = \int_{\partial D} P\,dx + Q\,dy$, if we chose

P, Q w/ $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

chose $Q=0$, $P=-y$; $\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} = 0+1 = \boxed{1}$

$\text{Area}(D) = \int_{\partial D} -y\,dx + 0\,dy = \int -y\,dx$

need parameters

The ellipse ∂D is parameterized by

$\vec{r}(t) = \langle a\cos(t), b\sin(t)\rangle$ on

$0 \le t \le 2\pi$ $\quad dx = x'(t)\,dt$

$-\int_{t_0}^{2\pi} b\sin(t)\,(-a\sin(t))$

$= ab\int_{t=0}^{2\pi} \sin^2(t)\,dt$

$= ab\int_0^{2\pi} \frac{1}{2}(1-\cos(2t))\,dt$

$= \frac{1}{2}ab\left[t - \frac{1}{2}\sin(2t)\right]_{t=0}^{2\pi} = \frac{1}{2}ab(2\pi-0)$

$= \boxed{ab\pi}$

§16.5 Curl and Divergence

Goal: define and study operations on vector fields

Defn: The curl of a vector field $\vec{v}$ on $\mathbb{R}^3$ (exactly 3 components) is

$v = \langle P, Q, R \rangle$

$$\underline{\text{Curl}}(\vec{v}) = \nabla \times \vec{v} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle$$

$$* \quad = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

ex: compute curl($\vec{v}$) for $\vec{v} = \langle xy, xyz, -y^2 \rangle$

Sol: $\text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xyz & -y^2 \end{vmatrix}$

*just a cross product

$= \langle \frac{d}{dy}[-y^2] - \frac{d}{dy}[xyz], \frac{d}{dz}[xy] - \frac{d}{dx}[-y^2], \frac{d}{dx}[xyz] - \frac{d}{dy}[xy] \rangle$

$= \langle -2y - xy, 0-0, yz - x \rangle$

$= \boxed{\langle -xy - 2y, 0, yz - x \rangle}$

Observe: Suppose we had a conservative VF $\vec{v} = \langle f_x, f_y, f_z \rangle$

So $\text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ d/dx & d/dy & d/dz \\ f_x & f_y & f_z \end{bmatrix}$

$= \langle \frac{df_z}{dy} - \frac{df_y}{dz}, \frac{df_x}{dz} - \frac{df_z}{dx}, \frac{df_y}{dx} - \frac{df_x}{dy} \rangle$

via Clairuts thm:

$\langle \underbrace{f_{zy} - f_{yz}}_{=}, \underbrace{f_{xz} - f_{zx}}_{=}, \underbrace{f_{yx} - f_{xy}}_{=} \rangle$

therefore $\boxed{\vec{0}}$

$\boxed{\text{curl}(\nabla f) = \vec{0}}$

* curl of conservative vector field is zero

*Proposition: A vf $\vec{v}$, comps having cts partials is conservative iff $\text{curl}(\vec{v}) = \vec{0}$

Defn: The divergence of a vf $\vec{v} = \langle v_1, \dots, v_n \rangle$ is

$\text{div}(\vec{v}) = \nabla \cdot \vec{v} = \langle \frac{d}{dx_1}, \dots, \frac{d}{x_n} \rangle \cdot \langle v_1, v_2, \dots \rangle$

$$= \sum_{i=1}^{n} \frac{dv_i}{dx_i}$$

ex: for $\vec{v} = \langle xy, xyz, -y^2 \rangle$

$$\text{div}(\vec{v}) = \frac{d}{dx}[xy] + \frac{d}{dy}[xyz] + \frac{d}{dz}[-y^2]$$

$$= y + xz + 0 = \boxed{y + xz}$$

Suppose: $\vec{v} = \text{curl}(\vec{w}), \vec{w} = \langle P, Q, R \rangle$

$$\vec{v} = \langle R_y - Q_z, -R_x - P_z, Q_x - P_y \rangle$$

$$\text{div}(\vec{v}) = \frac{d}{dx}[R_y - Q_z] + \frac{d}{dy}[P_z - R_x] + \frac{d}{dz}[Q_x - P_y]$$

$$= R_{yx} - Q_{zx} - R_{xy} + P_{zy} + Q_{xz} - P_{yz}$$

$$= (P_{zy} - P_{yz}) + (Q_{xy} - Q_{zx}) + (R_{yz} - R_{xy})$$

via clairuts

$$= 0 + 0 + 0 = \boxed{0}$$

* # not vector

* - a vf is the curl of another vf iff its divergence is zero

↳ above, showed $\text{div}(\text{curl}(\vec{v})) = 0$